#1: Answer key

Economics 630: Mathematical Economics, fall 2003
Professor Rebecca Menes
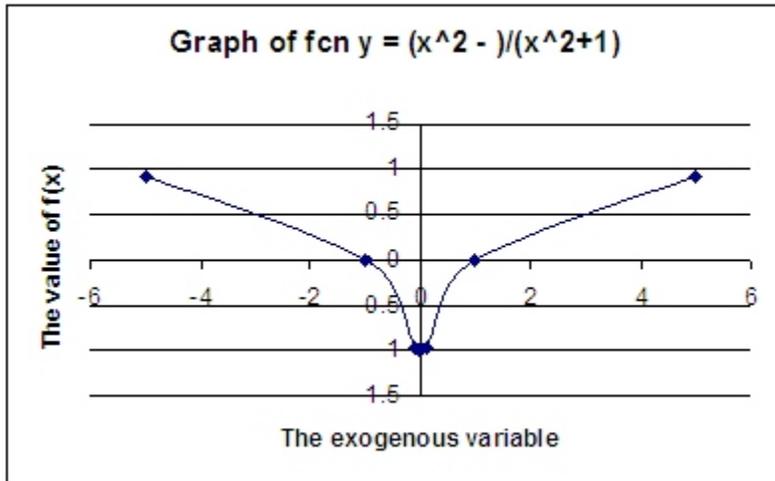rmenes@gmu.edu 703-993-1156

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1.A. Sketch the graph of the function $y = (x^2 - 1)/(x^2 + 1)$.

Test Points

x	y	
0	-1	
∞	1	fcn approaches 1, never gets there.
-∞	1	
1, -1	0	Fcn same for + and - values
2, -2	3/5	
3, -3	4/5	
4, -4	15/17	fcn is approaching 1, as predicted above

Graph of fcn y = (x^2 -)/(x^2+1)

Comments: Is the function differentiable at its critical point? Yes, so make sure the graph is "smooth" at that point. Does the function have horizontal asymptote? Yes: y = 1.

Often evaluating 3 types of points: the y-intercept, x = 0, and the extremum, x = ∞ and x = -∞, will give a quick insight into the function.

B. What is the domain of the function? What is the range?

Domain is the Rational Numbers, $(-∞,∞)$; $D \to R^1$.

> Comment: Although this is a rational function, the denominator never equals zero for any rational value of X, so we don't have any "gaps" in the domain.

Range is the interval $[-1, 1)$; $-1 \le y < 1$.

> Comment: The Range is closed at -1, but open at 1, the function never quite reaches 1.

2. Exercise 2.9,
b. G(x) is the cost of purchasing x units of some commodity.

> G'(x) is the cost of buying one more unit of x, or the Marginal Cost, which is USUALLY the PRICE!

c. H(p) is the amount of the commodity consumed when it's price is p.

> H(p) is a DEMAND function, and H'(p) is the increase in consumption when the price increases by one unit, that is the marginal demand with respect to price. Note that H'(p) * (p/H(p)) = price elasticity of demand.

e. S(Y) is the total national savings when national income (GDP) is Y.

> S(Y) is part of the National Income Accounts or the Circular Flow. The first derivative, S'(Y) is the change in the savings rate when income increases by one unit, (formally titled the Marginal Propensity to Save.)

3. Exercise 2.13 Prove parts A and B of theorem 2.4

A. $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$

Plan — Use the definition of the first derivative to set up both halves of the equality. Use algebraic manipulation to prove they are equal. I will prove for + and will claim that the same argument holds for -, by symmetry (although not wrong to prove it out again, either). So A simplifies to trying to prove the following:

$$\lim_{h \to 0} \frac{(f+g)(x_0+h) - (f+g)(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} + \lim_{h \to 0} \frac{g(x_0+h) - g(x_0)}{h}$$

Proof:

$(f+g)(x) = f(x) + g(x)$ Definition of the sum of two functions

Therefore the first derivative of (f+g) is:

$$\lim_{h \to 0} \frac{(f+g)(x_0+h) - (f+g)(x_0)}{h} = \lim_{h \to 0} \frac{[f(x_0+h) + g(x_0+h)] - [f(x_0) + g(x_0)]}{h}$$

And since the lim (A + B) = lim A + lim B   where A, B = fcns (sum of the limits is limit of the sums)

$$\lim_{h \to 0} \frac{(f(x_0+h)+g(x_0+h)) - (f(x_0)+g(x_0))}{h} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} + \lim_{h \to 0} \frac{g(x_0+h) - g(x_0)}{h}$$

And I hope you can see that the right hand side IS $f'(x_0) + g'(x_0)$ and so our theorem is proven.

Can repeat all steps to prove for $(f - g)' = f' - g'$

b. Prove $(kf)'(x_0) = k(f'(x_0))$

Plan — use the definition again.

$$\lim_{h \to 0} \frac{(kf(x_0+h) - kf(x_0))}{h} \qquad = \qquad k * \lim_{h \to 0} \frac{(f(x_0+h)) - f(x_0))}{h}$$

Constants are very obliging, and we can move them out of the parentheses, and out of limit entirely, so that the expression on the left hand side of the equals sign (the l.h.s.)

$$= \lim_{h \to 0} \frac{k(f(x_0+h) - f(x_0))}{h}$$

and moving the k again, through the limit operator (since limit of a constant = the constant)

$$= \qquad k * \lim_{h \to 0} \frac{(f(x_0+h)) - f(x_0))}{h} \qquad \text{and we are done.}$$

4. Find derivatives for the following (Comment - I don't care how much you simplify. In a real economics model you would simplify to the extent that the resulting expression was easy for your readers to understand.)

A. $f(x) = x^9 - 13x^3$ $\qquad\qquad\qquad\qquad$ $f'(x) = 9x^8 - 39x^2$

B. $f(x) = (x^3 - 1)/(x^3 + 1)$
Quotient Rule: $\qquad\qquad\qquad f'(x) = \dfrac{[(3x^2)(x^3 + 1) - (x^3 - 1)(3x^2)]}{(x^3+1)^2}$

$$= 6x^2 / (x^3 + 1)^2$$

C. $f(x) = (x^7 + x^5)(x^4 + 9)$
Could use the Product Rule, or could multiply out the polynomial and use the sum rule. I will use the Product Rule, but I checked my result by multiplying out the polynomial and using the sum rule.

$$f'(x) = [(7x^6 + 5 x^4)(x^4 + 9) + (x^7 + x^5)(4x^3)] = 11x^{10} + 9x^8 + 63x^6 + 45x^4$$

5. Exercise 2.17 (p. 33)
a. The function is both continuous and differentiable at all points, including x=0. The first derivative is $dy/dx = 2x$ if $x \geq 0$ and $dy/dx = -2x$ if $x<0$, which is itself continuous at all points. Therefore f(x) is C1, (continuously differentiable.)

b. The function is not continuous at $x = 0$, therefore NOT C1.

c. The function is continuous, but not differentiable at $x = 1$ (Compute the first derivative; there are two candidates for the derivative at x=1, depending on whether one evaluates the limit from "above" or "below" x=1. Since limit is not unique, derivative is not defined at x=1)

d. The function is continuous and differentiable, and the first derivative is continuous. Therefore function is C1. (Aside — note that the function is NOT C2, the first derivative is continuous, but it ISN'T differentiable at x = 1)

6. Exercise 2.18 on page 33.
Sketch the graph of the function $f(x) = x^{2/3}$

Comments:
This is a tricky one. Need to know how to read a fraction in the exponent: $x^{2/3}$ = the square of the cube root of x, or $(x^{1/3})^2$.

Domain for this function is $(-\infty,\infty)$. In the first answer key it was incorrectly identified at $[0, \infty)$. The inverse power 1/3 works fine for negative numbers. $(-9)^{1/3} = -3$.

Some points:

x	y	
0	0	
1	1	
8	4	(hint: start with 2, find 2^3 =8, plug in x = 8)
-1	-1	
∞	∞	
27	9	(hint: start with 3, find cube, plug it in)

Plot of the function f = x^(2/3)
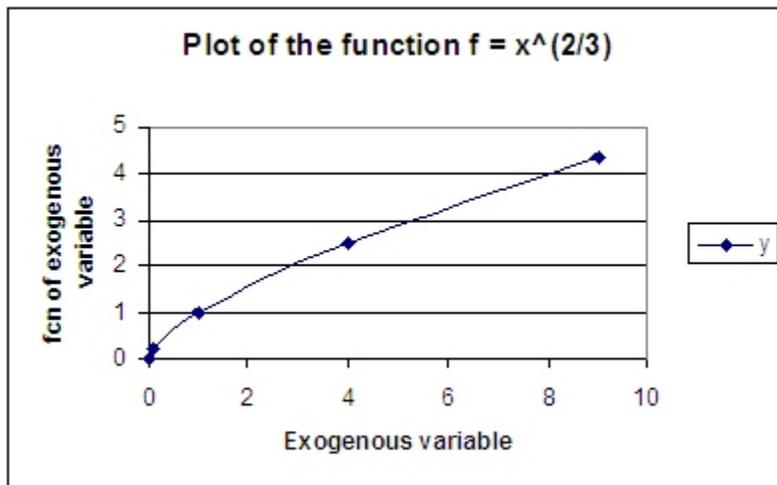
b. Describe the continuity and differentiability. It is continuous over entire domain, but not differentiable. First drv = $(2/3)*(x^{-1/3})$, which does not exist for x = 0.

The graph is SMOOTH and has NO BREAKS from 0 to ∞, but it has no derivative at the endpoint of 0.

7. Acme Widget Company has the following cost function: $Cost = 12 + 3x^2 + 7x$
And faces a market price of $25.00, where x = number of widgets.

A. The optimal output is found where MC = MR.
The MR = Market Price = 25.
$MC = dC/dx = 6x + 7 = 25$ at optimum
Solve for x: x=3. q.e.d.

B. linear approximation (differential approximation) = first derivative = MC = 25.
The exact increase:
$Cost(3) = 12 + 3*9 + 7*3 = 60$
$Cost(4) = 12 + 3*16 + 7*4 = 88$

$Cost(4) - Cost(3) = 88 - 60 = 28.$

Comment: Since Cost is a non-linear function with a positive second derivative, the linear approximation slightly underestimates the true marginal cost of increasing production from 3 to 4. That is, the linear approximation indicates firm would be indifferent between producing 3 and producing 4, but in fact the firm can be expected to stop at 3.