Answer #3

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Economics 630: Mathematical Economics I

Professor Rebecca Menes

rmenes@gmu.edu 703-993-1156

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1. Evaluate the following (1 point)
a) $2^5 = 2*2*2*2*2 = 32$ b) $4^{3/2} = (4*4*4)^{½} = 2*2*2 = 8$

2. Solve the following equations for x (3 points, based on 5.5)
a) $4e^{6x} = 24$ b) $\ln x^3 = 12$ c) $\ln x^{5/2} - 0.5 \ln x = \ln 25$

a) $\ln(4) + 6x = \ln(24)$; $x \approx (3.2 - 1.4) \div 6$; $x \approx 0.3$
b) $x = e^4 \approx 54.6$
c) Let $Z = \ln x$; $2z = \ln 25 \approx 3.22$; $z \approx 1.61$

3. Compute the first and second derivatives. (3 points, based on 5.8)

a) $f(x) = 2xe^{-7x}$; $df/dx = (2\text{-}14x)e^{-7x}$; hint: Use the product rule and the chain rule
b) $f(x) = \ln[(x^2 + 1)(x^2 - 1)]$; $f(x) = \ln(x^4 - 1)$; $df/dx = (4x^3) \div (x^4 - 1)$
c) $f(x) = [\ln(x)] \div x$; $df/dx = [(1/x)(x) - 1*\ln x] / x^2 = (1 - \ln x) / x^2$; hint: use the quotient rule.

4. Problem 5.10: Prove that if $y = \text{Log } x$, then $dy/dx = 1/(x \ln 10)$.
(3 points) Log x = log to the base 10.

Proof: The method used in lemma 5.3 is to invert the function, take the natural log and then differentiate.

$y = \text{Log } x$ implies $x = 10^y$
Therefore $\ln(x) = y \ln 10$; which can be re-arranged to give $y = \ln(x) / \ln(10)$
The rule for differentiating $\ln(x)$ yields $dy/dx = 1/(x * \ln(10))$. q.e.d.

5. Problem 5.11 (3 points)
At 10 percent annual interest rate, which of the following has the largest present value:
a) \$215 two years from now: $PV = 215\, e^{-0.1*2} \approx 215*0.819 = 176.09$
b) \$100 after each of the next two years: $PV = 100\, (e^{-0.1} + e^{-0.1*2}) \approx 100\, (0.905+0.819) = 172.40$
c) \$100 now and \$95 two years from now: $PV = 100 + 95(e^{-0.1*2}) \approx 100 + 95*0.819 = 177.81$
Therefore c) has the largest Present Value.

6. Problem 5.17 (3 points)
Prove that the elasticity of the product of two functions is the sum of the elasticities.
Remember that elasticity of f(x) wrt x = (df/dx)*(x/f(x))
Let f(x) = g(x)*h(x)
ln(f(x)) = ln((g(x)*h(x)) = ln g(x) + ln h(x)
d(ln f(x))/dx = d(ln g(x))/dx + d(ln h(x))/dx
(df/dx)*(1/f(x)) = (dg/dx)*(1/g(x)) + (dh/dx)*(1/h(x))
Multiply both sides by x and we get the elasticities:
(df/dx)*(x/f(x)) = (dg/dx)*(x/g(x)) + (dh/dx)*(x/h(x))

7. Today the American Economy is about 8 trillion dollars. If we maintain a growth rate of roughly 3% per year for the 21st century, roughly how big will the U.S. economy be in 2100? (1 point)

GNP(2100) = GNP(2000) * $e^{0.03*100}$ ≈ 8 trillion * 20.1 = 160.8 trillion

Or could use the "rule of 69", the economy will double every 69/3 = 23 years, which means roughly four times. 8*2*2*2*2 = 128 trillion in 92 years. In the remaining 8 years will pick up about another 25% growth (8 is about 1/3 of 23, but since we have compounded growth, we know that means 8 years will be less than 33% growth), which gets us to about 160 trillion.

8. A. Pbm 5.13: Suppose that you own a rare book whose value at time t years from now will be:

B(t) = 2^($t^{1/2}$) dollars.

Assuming a constant interest rate of 5 percent, when is the best time to sell the book and invest the proceeds?

The key insight is realizing that we are looking for t such that (dB/dt)/B(t) = 0.05, that is – in what year t is the change in the value of the book = 5%.

(dB/dt)/B(t) is found using the same trick we used when deriving the rule of 69 and in problem 4, taking the natural log and differentiating:

ln(B(t)) = $t^{0.5}$ ln(2)
d(ln(B(t)))/dt = (dB/dt)/B(t)) = ln(2)/2 * $t^{-0.5}$ = 0.05

and now we solve for t: $t^{1/2}$ = ln(2)/0.1 ≈ 6.9315; t = 48.04

B) Now assume that the bank interest rate rises to 7%. Now when is the best time to sell the painting and put the money in the bank. (2 points)

ln(2)/2 * $t^{-0.5}$ = 0.07 $t^{1/2}$ = 0.69315/0.14 ≈ 4.9511; t ≈ 24.5 years.

Notice how the relatively small increase in the interest rate cut the optimal holding time nearly in half.