Name: Econ 630, Fall 2003. Final

e-mail address:_________________________________
Phone number:_________________________________

General Instructions:

READ EACH QUESTION CAREFULLY. PARTIAL CREDIT WILL NOT BE GIVEN TO STUDENTS WHO FAIL TO GIVE THE CORRECT ANSWER BECAUSE THEY MISREAD THE QUESTION.

PUT YOUR NAME ON ALL OF THE SHEETS OF THE EXAM BOOKLET, INCLUDING THE SCRATCH PAPER AND THE GRAPH PAPER. DO ALL WORK IN THE EXAM BOOKLET.

CALCULATORS, SLIDE RULES, PENS, PENCILS, ERASERS, AND RULERS ARE THE ONLY TOOLS YOU MAY USE ON THIS EXAM.

PLEASE READ AND SIGN THE FOLLOWING:
I understand that academic integrity is highly valued at GMU. Further, I understand that academic dishonesty, such as cheating and plagiarism, are violations of University policy and will be pursued by the appropriate campus administrator. Finally, my signature below signifies that the work included is my own, and that I completed this assignment honestly.

Signature:____

Sanctions for academic dishonesty include suspension or dismissal from the university. There are alternatives to academic dishonesty. Please see you professor, advisor, or the Dean of Students to discuss other choices.

Point distribution:

Short Answers: 200 points	_______
Long Problem: 300 points	_______
Proof: 25 points extra credit	_______
Total	________

And best of luck!

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I. Short Answer (200 points. 15 questions)

1. Given the production function $q = AL^{\alpha}K^{(1-\alpha)}$

where L = labor employed in producing q

K = capital employed in producing q

Find the elasticity of production with respect to Labor: **ANS:** $\boldsymbol{\epsilon_{qL} = \alpha}$

Proof: I will use the fact that $\partial(\ln f(x)) / \partial(\ln x)$ = the elasticity

$\ln q = \ln A + \alpha \ln L + (1 - \alpha) \ln K$

$\partial(\ln q) / \partial(\ln L) = \alpha$

Therefore $\epsilon_{qL} = \alpha$

Remember: $\partial(\ln q) / \partial(\ln L) = \partial(\ln q)/\partial L * \partial L/\partial(\ln L) = (\partial q/\partial L)*(L/Q) = \epsilon_{qL}$

b/c $\partial L/\partial(\ln L) = L$

$\partial(\ln q)/\partial L = (\partial q/\partial L)/Q$

2. For a square NxN matrix, A, which of the following statements is NOT equivalent to the others:

a) A is invertible

b) Every system A**x** = **b** has at LEAST one solution for every **b**. (**x** and **b** are N dimensional vectors.)

c) Every system A**x** = **b** has at MOST one solution for every **b**. (**x** and **b** are N dimensional vectors.)

✔**d) A is singular.** This is the statement that doesn't fit. The other 4 statements all mean that A is NON singular

e) A has maximal rank of N.

3. What is the length of the vector **v** = (1, 1, 1, 1)? Answer: 2

$\| v \| = (1^2 + 1^2 + 1^2 + 1^2)^{1/2} = 2$

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4. Which of the following sets of vectors is a BASIS of R^3?

A) (1, 1, 1), (0, 0, 0) and (1, 0 , 0) (0,0,0) cannot be part of a spanning set

✔**B) (1, 0, 0), (0, 1, 0) and (3, 7, 13)** Three linearly independent 3-d vectors span R^3

C) (1, 0, 0), (2, 0, 0) and (3, 0, 0) NOT independent

D) (1, 0), (0, 1) and (1, 1) Not in 3 dimensions

E) (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0) and (0, 0, 0, 1) In 4 dimensions, one too many

5. Does the point (2, 4, 4) lies on the plane (x, y, z) = (1,2,3) + t (1,1,0) + s (0,1,1)?

Yes. s = 1, t =1 will return the point (2, 4, 4)

6. Find the first derivative of the following function:

$$y = (x^2 + 3x + 1)^5$$

$$dy/dx = 5 * (x^2 + 3x + 1)^4 (2x + 3)$$

7. How long does it take an economy to double in size if it is growing at the continuous rate of 3.5% per year?

The rule of 69: 0.69/0.035 = 19.7
To derive, solve the following for t:
$2 = 1 * e^{rt}$; $\ln 2 = rt$; $t = \ln 2/r \approx 0.69/0.035$

8. Given the quadratic equation: $x^2 + y^2 + z^2 = 0$

There exists some symmetric matrix, A, and vector $\mathbf{x}^T = (x, y, z)$ such that

$$\mathbf{x}^\mathbf{T}\ \mathbf{A}\ \mathbf{x} = x^2 + y^2 + z^2 = 0$$

Find the matrix A for which the above equation is true?

The matrix is the identity matrix,

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

How to see this? No terms which contain two different variables, so all the off-diagonal entries must be 0. The diagonals = the coefficients of the square terms = 1.

9. Find the first and second derivatives for the function $f(x) = (\ln x)^2$
The power rule: $d(g^2)/dx = 2g * dg/dx$

$df/dx = 2(\ln x) * (1/x) = (2 \ln(x)/x)$

$d^2f/dx^2 = -(2x^{-2})(\ln x) + 2x^{-2} = (2 - 2 \ln x)/x^2$

10. Solve the following constrained optimization (find x*, y* that maximize f(x,y):

Maximize $f(x, y) = \ln(x) + \ln(y)$

subject to the constraint that $x + 2y = 3$

$\mathcal{L} = \ln(x) + \ln(y) + \lambda(3 - x - 2y)$

f.o.c. $\partial\mathcal{L}/\partial x = 1/x - \lambda = 0$

$\partial\mathcal{L}/\partial y = 1/y - 2\lambda = 0$

$\partial\mathcal{L}/\partial\lambda = (3 - x - 2y) = 0$

$x/y = 2$; $x = 2y$; $(2y + 2y) = 3$; $y = 3/4$, $x = 1.5$

Extra Credit (5 points) : What is the value of λ?

$\lambda = 1/x = 2/3$

11. What is the Hessian (matrix of second partials) for the following function?

$f(x, y) = \ln(x) + \ln(y)$

First partials: $f_x = 1/x$

$f_y = 1/y$

$$H = \begin{matrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{matrix} = \begin{matrix} -x^{-2} & 0 \\ 0 & -y^{-2} \end{matrix}$$

12. The economy of Mason Land exports only Wizmoes. The quantity of Wizmoes exported, X, is increasing by 4% each year. The price of Wizmoes, P, is growing by 2% each year. What is the annual rate of growth of export revenue? (Hint – Export Revenue = P * X)

A) 4%

B) 2%

✔C) 6%

D) 8%

E) Cannot determine without knowing the total GDP of Mason Land.

To solve this one, you need to find rate of growth in revenue:

Remember, for any fcn y, rate of growth = (dy/dt)/y, and d(ln y(t))/dt = (dy/dt)/y
let R = Revenue from exports.

$R = PX$; $(dR/dt)/R$ is the percent change in R

$\ln R = \ln P + \ln X$

$d(\ln R)/dt = (dR/dt)/R = (dP/dt)/P + (dX/dt)/X = 4\% + 2\% = 6\%$

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Consider the Wizmo industry. Each firm faces the the Total Cost function for the production of Wizmoes:

$$TC = q^3 - 10q^2 + 100q$$

13. What is the quantity of Wizmoes, q, which minimizes the per Wizmo cost of production for a single firm (the most efficient scale?)

$ATC = q^2 - 10q + 100$

f.o.c. $2q - 10 = 0$ $q = 5$

14. If there is free entry and exit in the Wizmo industry, what is the long run market price of Wizmoes?

$MC = 3q^2 - 20q + 100$

At $q = 5$, $MC = 3*25 - 100 + 100 = \$75$

15) For the following system of equations:

$$x + y + z = 3$$
$$-x + y + 2z = 2$$
$$2x - 3y + z = 0$$

Set up the system in matrix form, Ax = b, and show that the matrix A is non-singular.

Find the values of (x, y, z) which solve for b = (3, 2, 0)

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 2 & -3 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix}$$

You can substitute in the original equation, or solve this system using Cramer's rule, but any way you work it, x = 1, y = 1, z = 1 is the solution of this system when b = (3, 2, 0)

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Part II: Pick three of the following four questions. DO NOT DO ALL FOUR QUESTIONS. IF YOU DO FOUR QUESTIONS I WILL ONLY GIVE CREDIT FOR YOUR THREE LOWEST!!!!!

Long Problem 1 "E-Gadds"

Gadd Inc. (now the 5th largest producer of gadgets in Ohio) adopts a new production process on the line at their plant in Akron. As the employees learn to handle the new process, production per hour rises at a rate of increase of 2% per per hour for every hour in operation.

A. Assuming Gadd Inc started with an output of 100 gadgets/hour at 9 am on Monday morning, April 8, 2002, what do you predict will be the output 40 hours later, at closing time on Friday, April 12, 2002? (Assume productivity compounds continuously.)

$A(40) = 100e^{0.02*40} = 100*e^{0.8} \approx 223$ gadgets/hour

B. How many hours will it take production to double?

Use the rule of 69: $69/2 \approx 34.5$ hours.

C. Gadd Inc's competitor, Acme Gadget, is also improving their production process. If Acme starts the week with an output of 90 gadgets per hour and ends the week (40 work-hours later) with an output of 134 gadgets per hour, what is the average percentage increase per hour in productivity per hour?

$135 = 90 * e^{40r}$

$\ln(135) = \ln(90) + 40r$

$\ln(135/90) = \ln(1.5) = 40r \approx .405$

Therefore $r \approx 1$

D. (Something completely different, but I needed to test another concept.)
You own a fine cheese whose value is given by the formula:

$$C(t) = (3/2)^{t\wedge(1/2)}$$

Where t is in years. If the interest rate is currently 10%. When should you sell your cheese.

Looking for the time at which (dC/dt)/C(t) = 10% We will use the trick that d(ln C(t))/dt = (dC/dt)/C(t)

$\ln(C(t)) = t^{1/2} \ln(3/2)$

$(dC/dt)/C(t) = d(\ln(C(t))/dt = \ln(3/2) /(2t^{1/2}) = 0.10$ solve for t

$\ln(3/2) \approx 0.406$

Therefore $0.203 \approx 0.1t^{1/2}$; $t^{1/2} \approx 2$; $t \approx 4$ You want to sell in approximately 4 years.

Long Problem 2

Consider the problem facing a revenue maximizing kleptocrat[1] choosing the level of taxes on the sale of Gadgets.

Assume the long run supply curve for Gadgets is infinitely elastic at a price of $1 per gadget. (That is, the long run marginal cost of gadgets, d(Cost)/dq = 1) This is an important assumption, because it means in the long run the supply curve is FLAT, the price that firms take home is always driven to $1.00 per unit.

Assume the Demand curve for gadgets is given by:

$$Q_d = 100 - 10\ P$$

A) What is the long run market equilibrium price and quantity in the ABSENCE of taxes.

The long run P* = MC = $1.00, the Q* = 90.

B) What is the long run market equilibrium if the kleptocrat imposes a tax of $1.00 on each Gadget. (The equilibrium values you must find are: the price paid by consumers, the price received by firms, the tax per unit going to the government, the total number of units traded, and the total revenue collected by the government.)

If government imposes a tax = $1.00, in the long run firms will exit the industry until firm take home price = MC = $1.00, so in the long run price paid by consumers = $1.00 + tax = $2.00.

The government keeps $1.00. The total number of units traded, Q, = 80, and the government gets $80.00 in revenue.

[1]Kleptocrat = a government official who uses the power of the government to maximize the total "profit" (revenue – costs) of government.

C) What is the tax that MAXIMIZES the revenue raised by the government? What is the MAXIMUM REVENUE the government takes home and what is the total quantity traded in the market at this tax rate?

Tax Revenue = Q * t = Q_d = (100 – 10 P)*t

But the P = Payment to firms + tax = ($1.00 + t)

So Tax Revenue = Qt = (100 – 10(1 + t))*t = 100t – 10t – $10t^2$ = 90t – $10t^2$

Max Tax Revenue: f.o.c. d(TR)/dt = 0
90 - 20t = 0
t = 4.5

Therefore the revenue maximizing tax = $4.50. At this tax level, the total Demand is

Q = 100 – 55 = 45 units

And the government collects 45*4.5 = $202.50

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Long Problem 3

A firm has a production function $q(K,L) = A K^{\alpha} L^{1-\alpha}$ and unit capital and labor costs of r and w, respectively. The firm can spend no more than C per week on inputs L and K.

A) Find K* and L* which maximize q for fixed C, expressing K* and L* in terms of w, r, C and α.

$\mathcal{L} = A K^{\alpha} L^{1-\alpha} + \lambda(C - wL - rK)$

f.o.c. $\partial\mathcal{L}/\partial K = A\alpha (L/K)^{1-\alpha} - \lambda r = 0$

$\partial\mathcal{L}/\partial L = A(1-\alpha) (K/L)^{\alpha} - \lambda w = 0$

$(L/K) = ((1 - \alpha)/\alpha) (r/w)$ $\quad$ $L = K * ((1 - \alpha)/\alpha) (r/w)$

$K = L * (\alpha/(1 - \alpha)) (w/r)$

Substitute into the budget constraint $C = wL + rK$

$$C = w(K * ((1 - \alpha)/\alpha) (r/w)) + rK = rK/\alpha$$

and

$$C = wL + r(L * (\alpha/(1 - \alpha)) (w/r)) = wL/(1- \alpha)$$

And we can solve each of these equations for K and L respectively to yield:

$$K = \alpha C/r$$

$$L = (1 - \alpha)C/w$$

B) Find the maximum weekly output achievable, q*, produced by K* and L*. (Express q as a function of w, r, C, α, and A, but NOT K or L)

$q(K,L) = A K^{\alpha} L^{1-\alpha} = A(\alpha C/r)^{\alpha}((1 - \alpha)C/w)^{1-\alpha} = CA(\alpha/r)^{\alpha} ((1- \alpha)/w)^{1-\alpha}$

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Long Problem #4

A) A factory makes 2 goods, grommets and widgets.

To make $1 worth of grommets requires $0.20 worth of grommets and $0.1 worth of widgets and 1 worker/hour.

To make $1 worth of widgets requires requires $0.05 worth of grommets and $0.1 worth of widgets and 0.5 worker/hours.

Set up the linear system of Equations that describes this economy:

Let G = Total production of Grommets
Let W = Total production of Widgets

$G = C_G$ + Grommets used to make Grommets + Grommets used to make Widgets
$W = C_W$ + Widgets used to make Grommets + Widgets used to make Widgets
This gives me the following 3 equations:

$$G = C_G + 0.2G + 0.05W$$

$$W = C_w + 0.1G + 0.1W$$

$$L = G + 0.5W$$

Which I now re-arrange to give me the equations that describe the economy

Consumption Grommets	C_G	=	$0.8G - 0.05W$
Consumption Widgets	C_W	=	$-0.1G + 0.9W$
Labor Employed	L	=	$G + 0.5W$

B) If the firm is planning to sell $750 worth of grommets and $500 worth of widgets, what must be the total production of grommets, the total production of widgets, and the number of workers employed?

$$750 = 0.8G - 0.05W$$

$$500 = -0.1G + 0.9W$$

Solving two equations in two unknowns:

It is a bit messy, but straightforward. $G = 9W - 5000$;

$$W = 664.34$$

$$G = 9*664.34 - 5000 = 979.06$$

Therefore Labor Employed, $L = G + 0.5W = 664.34 + 489.53 = 1153.87$

Proof: extra credit of 25 points

Given that $y = e^{g(x)}$ and that $d(e^x)/dx = e^x$, prove that $dy/dx = e^{g(x)} * dg(x)/dx$

y is a composite function, f(g(x)).

The chain rule is used to find the derivatives of composite functions:

$dy/dx = dy/dg * dg/dx$

$dy/dg = e^{g(x)}$

$dg/dx = dg/dx$

Therefore $dy/dx = e^{g(x)} * dg(x)/dx$ q.e.d.