Answers to Problem Set #3 – the Lagrangian.

7. A Lagrangian with algebra and even some economics, too.
Suppose that a firm's production function is given by the Cobb-Douglas function

$$q = K^{\alpha}L^{\beta}, \qquad \text{where } \alpha, \beta > 0,$$

and the firm purchases L, K in competive markets for wage w and rental rate of capital v.

a. Show that cost minimization requires

$$vK/\alpha = wL/\beta.$$

Two Proofs: (Because there is more than one way to skin a cat)

Proof 1: Use a Lagrangian
Objective Function: Minimize Costs st the constraint of a fixed q:

$$\mathcal{L} = wL + vK - \lambda(q - K^{\alpha}L^{\beta})$$

f.o.c.

$$\partial\mathcal{L}/\partial L = w - \lambda(\beta K^{\alpha}L^{\beta-1}) = 0$$

$$\partial\mathcal{L}/\partial K = v - \lambda(\alpha K^{\alpha-1}L^{\beta}) = 0$$

Therefore $w/v = (\beta/\alpha)(K/L)$

Which we can rearrange to get $vK/\alpha = wL/\beta$

Proof 2: Show that this is the SAME as requiring that the marginal returns to L and K be proportional to the costs of the inputs, w and v.
Step #1:
Since w & v show up in our answer, this suggests we should start with a formula or relationship that involves w & v. The first thing I know about the relationship between the price of inputs and their productivity is the following: the ratio of the prices of the inputs = ratio of the physical marginal productivity.

w/v = Marginal Physical Product of Labor / Marginal Physical Product of Capital

Which is just the relationship between the costs of inputs and the first drv of the Production Function.

$$w/v = MPL / MPK = \partial Q/\partial L \,/\, \partial Q/\partial K$$

Now, in order for the proof to be complete I will have to prove this assertion – and for that I will need a Lagrangian. But before I do so, I will determine that this IS the relationship I need.

Step #2:
Okay, this looks good. Now return to the production function and find the partials:

$$\partial Q/\partial L = \beta K^{\alpha} L^{\beta - 1} \qquad \partial Q/\partial K = \alpha K^{\alpha - 1} L^{\beta}$$

Step #3:
Put the pieces together and see whether it works:

$$w/v = MPL / MPK = \frac{\beta K^{\alpha} L^{\beta - 1}}{\alpha K^{\alpha - 1} L^{\beta}}$$

And now its algebra:

$$w/v = \frac{\beta K}{\alpha L}$$ Which does indeed rearrange to: $wL/\beta = vK/\alpha$

b. Assuming cost minimization, show that total costs can be expressed as a function of q, v, w of the form:

TC = Bq^(1/(α+β))w^(β/(α+β))v^(α/(α+β)),

where B is a constant depending on α and β.
Hint: This part may be most easily worked by using the results from part (a) to solve successively for TC as a function of L and TC as a function of K and then substituting into the production function.

Apologies because the algebra for this one is really unpleasant, but let's forge ahead.

What we start with:
$wL/\beta = vK/\alpha$ & $TC = wL + vK$ & $q = K^{\alpha}L^{\beta}$

The hint tells us to solve the first equation for L and for K.

$K = (\alpha/\beta)(w/v)L$ $\qquad$ $L = (\beta/\alpha)(v/w)K$

$q = [(\alpha/\beta)(w/v)L]^{\alpha}L^{\beta}$ $\qquad$ $q = K^{\alpha}[(\beta/\alpha)(v/w)K]^{\beta}$

ugly algebra $\qquad$ ugly algebra

$L = q^{(1/(\alpha + \beta))} (\beta v/\alpha w)^{(\alpha/(\alpha+\beta))}$ $\qquad$ $K = q^{(1/(\alpha + \beta))} (\alpha w/\beta v)^{(\beta/(\alpha + \beta))}$

Substitute these two terms into our TC function, TC = wL + vK, then carry out some really, really ugly algebra to get:

$TC = q^{(1/(\alpha+\beta))} [(\beta v/\alpha w)^{(\alpha/(\alpha+\beta))} * w + (\alpha w/\beta v)^{(\beta/(\alpha+\beta))} * v]$

And then more ugly algebra to separate the α's and β's from the v's and w's yields

$TC = q^{(1/(\alpha+\beta))} [(\beta/\alpha)^{(\alpha/(\alpha+\beta))} * (v/w)^{(\alpha/(\alpha+\beta))} * w + (\alpha/\beta)^{(\beta/(\alpha+\beta))} (w/v)^{(\beta/(\alpha+\beta))} * v]$

$TC = q^{(1/(\alpha+\beta))} [(\beta/\alpha)^{(\alpha/(\alpha+\beta))} * v^{(\alpha/(\alpha+\beta))} * w^{(\beta/(\alpha+\beta)} + (\alpha/\beta)^{(\beta/(\alpha+\beta))} * w^{(\beta/(\alpha+\beta))} * v^{(\alpha/(\alpha+\beta)}]$

$TC = q^{(1/(\alpha+\beta))} [(\beta/\alpha)^{(\alpha/(\alpha+\beta))} + (\alpha/\beta)^{(\beta/(\alpha+\beta))}] v^{(\alpha/(\alpha+\beta))} * w^{(\beta/(\alpha+\beta)}$

Which is what we are looking for, if $B = [(\beta/\alpha)^{(\alpha/(\alpha+\beta))} + (\alpha/\beta)^{(\beta/(\alpha+\beta))}]$!

Apologies for the ugly, ugly algebra – but even very simple models can get messy.

c. Show that if α + β = 1, the TC is proportional to (linear in) q.

This one is a little gift. Note that it can be done, using the formula you are GIVEN for part b, even if you did not manage to prove b. Just rewrite the above function, but substituting 1 for α + β:

$TC = q\ [(\beta/\alpha)^{\alpha} + (\alpha/\beta)^{\beta}]\ v^{\alpha} * w^{\beta}$.

Since α, β, w, and v are NOT functions of q, the TC fcn is linear in q.

d. Calculate the firm's marginal cost curve. Show that

$\epsilon(MC \text{ wrt } w) = \beta / (\alpha + \beta)$ — elasticity of the MC with respect to the wage

$\epsilon(MC \text{ wrt } v) = \alpha / (\alpha + \beta)$ — elasticity of the MC with respect to the rental rate of capital.

Marginal Cost is just the first derivative of the TC, which is a power fcn of q:

$d(TC)/dq = (1 / (\alpha + \beta))\ q^{((1-\alpha-\beta)/(\alpha+\beta))} [(\beta/\alpha)^{(\alpha/(\alpha+\beta))} + (\alpha/\beta)^{(\beta/(\alpha+\beta))}]\ v^{(\alpha/(\alpha+\beta))} * w^{(\beta/(\alpha+\beta)}$

Notice how much stuff just comes along for the ride.

And now for the elasticities –

$\epsilon(MC \text{ wrt } w) = \partial(MC)/\partial w * w/MC$

We can make our life a lot easier if we recognize that a bunch of stuff is constant in w, so we can set it all equal to some constant C.

Let $C = (1 / (\alpha + \beta))\, q^{((1-\alpha-\beta)/(\alpha+\beta))} [(\beta/\alpha)^{(\alpha/(\alpha+\beta))} + (\alpha/\beta)^{(\beta/(\alpha+\beta))}]\, v^{(\alpha/(\alpha+\beta))}$

So we can rewrite $MC = C * w^{(\beta/(\alpha+\beta)}$

And therefore $\partial(MC)/\partial w = (\beta / (\alpha + \beta))\, C\, w^{(-\alpha/(\alpha+\beta))}$

And $\epsilon(MC \text{ wrt } w) = \partial(MC)/\partial w * w/MC = (\beta / (\alpha + \beta))\, C\, w^{(-\alpha/(\alpha+\beta))} * w/(C * w^{(\beta/(\alpha+\beta)})$

Which simplifies nicely to $\epsilon(MC \text{ wrt } w) = \partial(MC)/\partial w * w/MC = (\beta / (\alpha + \beta))$. Q.e.d.

The elasticity of the MC with respect to the rental rate of capital, $\epsilon(MC \text{ wrt } v) = \alpha / (\alpha + \beta)$, can be solved in a similar manner. Or we can note that the MC function is symmetric in w and v with the exception of the substitution of β for α, so the elasticity will differ only in swapping α and β.

And we are done.

This is an ugly problem. Not difficult, but ugly. It is not, however, an unusual amount of algebraic manipulation for a model of moderate sophistication. I find that when working on such a problem that I need lots of blank paper, and that it helps a lot to number the pages as I use them, so I can go back and retrace my work. Also I take notes as I work on what I think I am trying to do.