Problem Set #3 (Actually this was the fourth problem set handed in)
Econ 812: Section 002, Spring 2004

Schaums Chapters 10, 11 and 12, Varian Chapters 26 and 27. Simon & Blume Chapters 6 – 11. I recommend working through all of the Schaum's problems, then reading Varian 26 and 27. If you get a chance and you have it, look at Simon & Blume – it covers slightly different information. I will review some of the Simon and Blume material next week.

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Short Answer Questions:

1. For a square NxN matrix, A, which of the following statements is NOT equivalent to the others:

a) A is invertible
b) Every system A**x** = **b** has at LEAST one solution for every **b**. (**x** and **b** are N dimensional vectors.)
c) Every system A**x** = **b** has at MOST one solution for every **b**. (**x** and **b** are N dimensional vectors.)
d) A is singular. THIS IS THE STATEMENT NOT EQUIVALENT TO THE OTHERS
e) A has maximal rank of N.

2. What is the length of the vector **v** = (1, 1, 1, 1)?

Pythagorean theorem: $(1^2 + 1^2 + 1^2 + 1^2)^{½} = (4)^{½} = 2$

3. Which of the following sets of vectors is a BASIS of R^3?

A) (1, 1, 1), (0, 0, 0) and (1, 0 , 0) NO. (0,0,0) is not a useful basis vector.
B) (1, 0, 0), (0, 1, 0) and (3, 7, 13) YES.
C) (1, 0, 0), (2, 0, 0) and (3, 0, 0) NO. (2,0,0) and (3,0,0) are linearly dependent.
D) (1, 0), (0, 1) and (1, 1) NO. These are vectors in R^2
E) (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0) and (0, 0, 0, 1) NO. This is a spanning set in R^4.

4) Find a vector of length 1 that points in the same direction as the vector (1,1,1).

It will be a vector of the form (a, a, a) where $(a^2 + a^2 + a^2)^{½} =(3a^2)^{½} = 1$.

Solving for a, we get $a = (1/3)^{½}$, and our vector will be $((1/3)^{½}, (1/3)^{½}, (1/3)^{½})$

5) (Simon & Blume pbm 7.20) Compute the rank of the following matrix:

1	6	– 7	3
1	9	– 6	4
1	3	– 8	4

We have three rows, (three equations) so the maximum rank this matrix can have is 3. To show that the rank is indeed three we can take several paths. Perhaps the easiest is Gaussian elimination. We can't make any of the rows completely into zeros, so we do indeed have three independent rows, and the rank is three.

Try it – the end of Gaussian elimination is

1	6	– 7	3
0	3	1	1
0	0	0	1

6) (Simon & Blume p. 108-109) Consider a firm with profits = π. The company has agreed, as part of a class action settlement, to deposit 10% of its after tax profits to a trust fund established to pay product liability claims. It must pay a state tax of 5% of its profits, computed AFTER making the payment into the trust fund, and a federal tax of 40%, AFTER the trust fund and tax payments are made. How much does the company pay in state taxes, federal taxes, and payment to the trust fund, as a function of π?

A problem in setting up a set of linear equations:

Let R = deposit into trust fund
Let S = State taxes
Let F = Federal taxes

Step #1: translate the words into equations
$R = 0.10\ (\pi - S - F)$
$S = 0.05\ (\pi - R\ )$
$F = 0.40\ (\pi - R - S)$

Which rearranges to:
$\pi = 10\ R + S + F$
$\pi = R + 20\ S$
$\pi = R + S + 2.5\ F$

The algebra is messy, but the rest is just substitution.

7. Find the determinant (if it exists) and determine if positive or negative (semi) definite:

A)

1	2	3
3	3	3
2	0	0

$|A| = 1 * (3*0 - 3*0) - 2 * (3*0 - 3*2) + 3 * (3*0 - 3*2) = 0 + 12 - 18 = -6$

Principal Minors: $|H_1| = 1 > 0$
$|H_2| = 3 - 6 = -3 < 0$
$|H_3| = |A| = -6 < 0$

This is NEITHER positive nor negative (semi) definite.

B)

11	– 3
-45	12

$|A| = 11*12 - (-3)*(-45) = 132 - 135 = -3$

Principal Minors: $|H_1| = 11 > 0$
$|H_2| = |A| = -3 < 0$

This is NEITHER positive nor negative (semi) definite.

C)

1	0	0
3	11	0

NOT square, NO DETERMINANT, neither positive nor negative semi-definite.

8) Find the inverse:

2	3	2
1	0	0
8	9	9

Finding an inverse is a messy business – on the exam I would never have you find an inverse, but I might give you a selection of matrices and have you determine which two ARE inverses. (Multiply them together to get the identity matrix.)

Rule $A^{-1} = \text{Adj } A / |A|$

Minor = $|M_{ij}|$ = determinant of the submatrix found by deleting ith row and jth column

Cofactor = $(-1)^{i+j} |M_{ij}|$

Adj A = Transpose of the matrix of cofactors

So for our above matrix: Cofactor Matrix C =

0	– 9	9
– 9	2	6
0	2	– 3

And the Adj A =

0	– 9	0
– 9	2	2
9	6	– 3

Determinant of A = |A| = – 9

So the Inverse = Adj A / |A| =

0	1	0
1	– 2/9	– 2/9
– 1	– 2/3	1/3

I checked myself using the minverse function in excel.

9. Cramer's rule to solve the following:

$$3x + 5y = 13$$
$$9x + y = 11$$

Cramer's rule: $x_i = |A_i| / |A|$ where A_i = coefficient matrix with column i replaced by b.

A =

3	5
9	1

b =

13
11

A_1 =

13	5
11	1

A_2 =

3	13
9	11

$|A| = 3 – 45 = – 42$ $|A_1| = 13 – 55 = – 42$ $|A_2| = 33 – 121 = – 84$

$x = – 42/ – 42 = 1$
$y = – 84 / – 42 = 2$

Check: 3 + 10 = 13; 9 + 2 = 11. ✔

10. What is the Jacobian Matrix for the function in 9.
Jacobian = matrix of first partials. $f_1(x,y) = 3x + 5y$
$f_2(x,y) = 9x + y$

J =

$\partial f_1/\partial x$	$\partial f_1/\partial y$	=	3	5
$\partial f_2/\partial x$	$\partial f_2/\partial y$		9	1

The DETERMINANT of the Jacobian = 0 if the system is SINGULAR. (We already know it is NON-SINGULAR b/c Cramer's rule worked.) |J| = 3 – 45 = – 42. Same as the determinant for the underlying function, b/c this is a linear function.