Econ 812: Problem Set #5
Econ 812: Spring 2004

Covers: Schaums 10 - 12 and/or Simon & Blume 6 – 11. These two sources cover the material VERY differently, and the geometric material (vectors and planes) in only in Simon & Blume.

What you need to know:
1) How to set up a system of linear equations
2) When a solution exists and how to find it
3) Matrix algebra – including inverse, transpose, adjoint, elementary row operations, etc.
4) Determinants – how to find them and what they mean. Minors and cofactors, Hessian and Jacobian matrices. Quadratic equations and the discriminant.
5) Cramer's rule and Gaussian Elimination
6. Vectors and linear functions in $R^n$ (hyperplanes).

Problems
1) Does the point $(4,3,2)^T$ lies on the plane $(1,2,3)^T + t(1,1,0)^T + s(0,1,1)^T = (x,y,z)^T$

Do s and t exist such that $(1, 2, 3)^T + t(1, 1, 0)^T + s(0, 1, 1)^T = (4, 3, 2)^T$

| | | | |
|---|---|---|---|
| Decompose into: | $1 + t = 4$ | $t = 3$ | |
| | $2 + t + s = 3$ | $2 + 3 – 1 = 4$ | NO. |
| | $3 + s = 2$ | $s = – 1$ | |

The point does NOT lie on the plane.

2) Give the product vector (widgets, xylophones, zithers) and the Price vector (Pw, Px, Pz) = (2, 3, 1.5) and a budget of 15, write down the point normal equation for the budget constraint and multiply it out – to give the "standard" version.

$(2, 3, 1.5) \cdot (w, x, z) = 15$
$2w + 3x + 1.5z = 15$

3) Consider the following system: **A x = b**

| | | | |
|---|---|---|---|
| Where **A** | 8 | 3 | 2 |
| | 6 | 4 | 7 |
| | 5 | 1 | 3 |

a) find the determinant

$$8(12 – 7) – 3(18 – 35) + 2(6 – 20) = 8*5 + 3*17 – 2*14 = 40 + 51 – 28 = 63$$

b) find the adjoint of A
Adjoint of A = transpose of the vector of Cofactors.

Cofactor = $(-1)^{i+j} | M_{ij} |$     $| M_{ij} |$ = determinant of the sub-matrix created when row i and column j are eliminated.

Adjoint of A =

| | | |
|---|---|---|
| $C_{11} = (12 - 7) = 5$ | $C_{21} = (-1)(9 - 2) = -7$ | $C_{31} = (21 - 8) = 13$ |
| $C_{12} = (-1)(18 - 35) = 17$ | $C_{22} = (24 - 10) = 14$ | $C_{32} = (-1)(56 - 12) = -44$ |
| $C_{13} = (6 - 20) = -14$ | $C_{23} = (-1)(8 - 15) = 7$ | $C_{33} = (32 - 18) = 14$ |

c) Solve for $\mathbf{b} = (20, 35, 16)^T$ using Cramer's Rule. Show your work.

**A x = b**

Where **A**

$$\begin{matrix} 8 & 3 & 2 \\ 6 & 4 & 7 \\ 5 & 1 & 3 \end{matrix} \quad \begin{matrix} x \\ y \\ z \end{matrix} \quad = \quad \begin{matrix} 20 \\ 35 \\ 16 \end{matrix}$$

Cramer's rule: $x = | A_x | / | A |$

$y = | A_y | / | A |$

$z = | A_z | / | A |$

from above, | A | = 63.

$$A_x = \begin{matrix} 20 & 3 & 2 \\ 35 & 4 & 7 \\ 16 & 1 & 3 \end{matrix} \qquad |A_x| = 20(12 - 7) - 3(105 - 112) + 2(35 - 64) = $$
$$= 20*5 + 3*7 - 2*29 = 100 + 21 - 58 = 63$$

x = 1

$$A_y = \begin{matrix} 8 & 20 & 2 \\ 6 & 35 & 7 \\ 5 & 16 & 3 \end{matrix} \qquad | A_y | = 8(105 - 112) - 20(18 - 35) + 2(96 - 175)$$
$$= -8*7 + 20*17 - 2*79 = 126$$

x = 126/63 = 2

$$A_z = \begin{matrix} 8 & 3 & 20 \\ 6 & 4 & 35 \\ 5 & 1 & 16 \end{matrix} \qquad | A_z | = 8(64 - 35) - 3(96 - 175) + 20(6 - 20)$$
$$= 232 + 237 - 280 = 189$$

z = 189/63 = 3

d) Solve the system again, using Gaussian elimination. Again, show your work.

I am going to show Gaussian Elimination for a simpler system:

$2x + y = 4$
$3x + y = 5$

| | | | | | |
|---|---|---|---|---|---|
| A | = | 2 | 1 | b = | 4 |
| | | 3 | 1 | | 5 |

Gaussian elimination:

| | | | |
|---|---|---|---|
| 2 | 1 | 1 | 0 |
| 3 | 1 | 0 | 1 |

Multiply row 1 by 3/2 and subtract from row 2

| | | | |
|---|---|---|---|
| 2 | 1 | 1 | 0 |
| 3 – (3/2)2 | 1 – 3/2 | 0 – 3/2 | 1 – 0 |

| | | | |
|---|---|---|---|
| 2 | 1 | 1 | 0 |
| 0 | – ½ | – 3/2 | 1 |

Multiply row 2 by 2 and add to row 1

| | | | |
|---|---|---|---|
| 2 + 0 | 1 – 1 | 1 – 3 | 0 + 2 |
| 0 | – ½ | – 3/2 | 1 |

| | | | |
|---|---|---|---|
| 2 | 0 | – 2 | 2 |
| 0 | – ½ | – 3/2 | 1 |

Multiply the firest row by ½ and the second row by –2

| | | | |
|---|---|---|---|
| 1 | 0 | – 1 | 1 |
| 0 | 1 | 3 | – 2 |

$$A^{-1} = \begin{vmatrix} -1 & 1 \\ 3 & -2 \end{vmatrix}$$

So $A^{-1}$ **b** = **x**

$$\begin{vmatrix} -1 & 1 \\ 3 & -2 \end{vmatrix} \begin{vmatrix} 4 \\ 5 \end{vmatrix} = \begin{matrix} -4 + 5 = x & x = 1 \\ 12 - 10 = y & y = 2 \end{matrix}$$