Comments – This problem set develops the relationship between production functions and cost functions, which when combined with the input and output prices, allows us to go from production functions to supply functions and quantities.

1. Finding σ – the Elasticity of the (K/L) ratio with respect to the Rate of Technical Substitution.

This question is based on 1.8 and 1.9 in Varian.
First, it is worth re-defining and re-deriving the RTS. We are interested in how we trade off K for L along an isoquant – q fixed at some q*. That is, we are starting with a production function q* = f(K, L), and looking for RTS. The slope of the isoquant tells us how much K we must add if we subtract L, assuming we want to keep producing q*. That is, RTS = – dK/dL, where dK/dL = slope of the isoquant.

To get any further, we make use of the Total Differential (the Total Differential is often used to derive formulas which concern relationships of drvs).

Starting with $q^* = f(K, L)$

$dq^* = (\partial f/\partial K)\, dK + (\partial f/\partial L)\, dL = 0$ b/c q* = constant

Therefore $RTS = (-\, dK/dL) = (\partial f/\partial L) / (\partial f/\partial K)$

And now we have our formula – we need to find the partials of the production function and put them INTO this formula.

A. Given a Cobb Douglas Production Function: $q = A\, K^{\alpha}\, L^{\beta}$

Show that $\sigma = \partial \ln (K/L) / \partial \ln (RTS) = 1$

$q = A\, K^{\alpha}\, L^{\beta}$ $\quad \partial f / \partial L = \beta A K^{\alpha} L^{\beta - 1}$ $\quad \partial f / \partial K = \alpha A K^{\alpha - 1} L^{\beta}$

$RTS = -\, dK/dL = (\beta A K^{\alpha} L^{\beta - 1}) / (\alpha A K^{\alpha - 1} L^{\beta}) = (\beta / \alpha)(K / L)$

We are now half-way to our answer – but we need to EXPRESS (K/L) as a function of RTS in order to find the elasticity of (K / L) wrt RTS.

$(K / L) = (\alpha / \beta)(RTS)$ – A LINEAR function, so taking first drv is easy.

$d(K / L) / d(RTS) = (\alpha / \beta)$

And $\sigma = d(K / L) / d(RTS) * (RTS) / (K / L) = (\alpha / \beta)(\beta / \alpha)(K / L)(L / K) = 1$

B. Given a CES Production Function (Constant Elasticity of Substitution)

$q = f(K,L) = [K^{\rho} + L^{\rho}]^{\epsilon/\rho}$ $\quad \rho \leq 1,\ \rho \neq 0,\ \epsilon > 0.$

Show that $\sigma = 1 / (1 - \rho)$.

Same plan – find RTS as a fcn of (K/L), re-express (K/L) as a fcn of RTS, then find σ.

$\partial f/\partial L = (\epsilon / \rho)(K^{\rho} + L^{\rho})^{(\epsilon - \rho) / \rho} (\rho L^{\rho - 1})$

$\partial f/\partial K = (\epsilon / \rho)(K^{\rho} + L^{\rho})^{(\epsilon - \rho) / \rho} (\rho K^{\rho - 1})$

So $RTS = (\partial f / \partial L) / (\partial f / \partial K) = (L / K)^{(\rho - 1)}$

And we can write $(L / K) = (RTS)^{(1 / (\rho - 1))}$

Life is easier if we temporarily introduce some new notation: $a = (1 / (\rho - 1))$

$$(L / K) = (RTS)^{a}$$

$$d(L / K) / d(RTS) = a(RTS)^{a - 1}$$

And Elasticiy is $\sigma = d(L / K) / d(RTS) * (RTS) / (L / K) = a(RTS)^{a - 1} (RTS) (RTS)^{- a}$

Add up all the power: $a - 1 + 1 - a = 0$, so we have $\sigma = a (RTS)^{0} = a = (1 / (\rho - 1))$.

Q.e.d

2. Rework the example which we worked in lecture. We start with the production function for a Hamburger Village, a small restaurant:

Production function: $q = 10 K^{1/2} L^{1/2}$

where q = number of hamburgers
K = capital (in 1,000s of dollars)
L = Labor (in person-hours)
All per hour.

A. Find the formula for the isoquants q = 50, q = 100, and the general formula $q = q_0$.

$50 = 10 K^{1/2} L^{1/2}$	$5 = K^{1/2} L^{1/2}$	$25 = K L$	$K = 25 / L$
$100 = 10 K^{1/2} L^{1/2}$	$10 = K^{1/2} L^{1/2}$	$100 = K L$	$K = 100 / L$
$q_0 = 10 K^{1/2} L^{1/2}$	$(q_0 / 10) = K^{1/2} L^{1/2}$	$(q_0)^2 / 100 = K L$	$K = (q_0)^2 /(100 L)$

B. Graph the isoquants q = 50 and q = 100 (to scale, you will need graph paper)

C. Show that the Rate of Technical Substitution of L for K = K/L

$q = 10\ K^{1/2}\ L^{1/2}$
$RTS = (\partial f / \partial L) / (\partial f / \partial K) = 5(K/L)^{½} / 5(L/K)^{½} = K/L$

D. Show that cost minimization requires that w/v = K/L

Cost Minimization – the Lagrangian rears its ugly head.

Minimize TC st $q = q_0$

Lagrangian: $\mathcal{L} = wL + vK + \lambda(q_0 - 10\ K^{1/2}\ L^{1/2})$

First Order Conditions $\partial\mathcal{L}/\partial L = w - 5\ \lambda\ (K/L)^{½} = 0$

$\partial\mathcal{L}/\partial K = v - 5\ \lambda\ (L/K)^{½} = 0$

$w = 5\ \lambda\ (K/L)^{½}$

$v = 5\ \lambda\ (L/K)^{½}$

So we can solve for w/v = K/L

E. Find $MP_{K,}$ and MP_L, K* and L* if $q_0 = 40$, v = 12, w = 4.

$MP_L = (\partial f / \partial L) = 5(K/L)^{½} = 5(w/v)^{½} = 5(4/12)^{½} = 5 / (3)^{½} = 2.89$

$MP_K = (\partial f / \partial K) = 5(L/K)^{½} = 5(v/w)^{½} = 5(12/4)^{½} = 5(3)^{½} = 8.65$

F. Find the TC = f(q)

What we know:

TC = wL + vK	$q = 10(KL)^{½}$	and from our optimization: w/v = K/L
K = (w/v)L	$q = 10(w/v)^{½}\ L$	$L = (q / 10)(v / w)^{½}$
L = (v/w)K	$q = 10(v/w)^{½}\ K$	$K = (q / 10)(w / v)^{½}$

$TC = w (q / 10)(v / w)^{½} + v (q / 10)(w / v)^{½} = ((vw)^{½} / 5)q$

G. Show that the Short Run Supply Curve for this firm is $q_s = ((50 * K_1) / w) * P$
Where K_1 = the fixed quantity of capital with which the firm must work in the SR.
(Here we have to throw out our carefully developed optimal K/L ratio, since we have fixed K)

$TC = wL + vK_1$ $q^2 = (100K_1)L$ $L = q^2 / (100K_1)$

$STC = (w\ q^2 / (100K_1)) + vK_1$

The inverse supply curve is: $MC = d(TC) / dq = wq / (50K_1)) = \text{Price}$
So the Supply Curve is: $q = (50K_1 * P) / w$ q.e.d.

H. Show that for THIS function the "shut down" price = 0.

Shut down when Price doesn't cover Average Variable Costs. So we need to find the q for which MC < VC.

Variable Costs for this function = $wL = w*(q^2 / (100K_1))$, Avg VC = $(wq) / (100K_1)$
Price = MC = $wq / (50K_1))$

Therefore Price = 2*AVC for all q, never shut down.

I. Find the Short Run Profit Function as a function of w, v, K_1, and P. Then show that for $w = v = K_1 = 4$, the Short Run Profit Function is: $\pi(q) = P * q - 16 - q^2 / 100$

Short Run Profit Function = $STR(q) - STC(K_1 \mid q)$

$$SR\pi = Pq - (wq^2 / (100K_1)) - vK_1$$

$$Sr\pi(w = v = K_1 = 4) = Pq - (4q^2 / 400) - 4*4 = Pq - (q^2 / 100) - 16$$

J. Finally – find the minimum Short Run Cost Function. Show that if $v = w = K_1 = 4$, the minimum short run AVERAGE cost is \$0.80 (80 cents) per burger.

$$STC = (w\, q^2 / (100K_1)) + vK_1 = (q^2 / 100) + 16$$

$$SAC = STC/q = (q/100) + 16q^{-1}$$

$$d(SAC)/dq = (1/100) - 16/q^2 = 0 \qquad q^2 = 1600 \qquad q = 40$$

At q = 40, $SAC = (40/100) + (16/40) = 0.4 + 0.4 = 0.8$

K. DISCUSS the Long Run: What is the long run price? The long run profit? The long run number of firms?

In the Long Run the price = MC = AC = $(vw)^{1/2} / 5$. Long run profit = 0. Long run number of firms is not determined by this technology, since we have constant returns to scale.

3. Suppose that a firm's fixed proportion production function is given by $q = \min(5K, 10L)$

and that the rental rates of K and L are given by $v = 1$ and $w = 3$.

A) Calculate the firm's long run total, average, and marginal cost curves
Leontief production functions are also CRS.
To make one unit of q requires 0.2 units of K and 0.1 units of L; $1 = \min(5*0.2, 10*0.1)$
So the cost of one unit of production $= 0.2v + 0.1w = 0.2 + 0.3 = 0.5$
Therefore Long Run Avg Cost = Long Run Marginal Cost = 0.5
Long Run Total Cost = 0.5q

B) Suppose that K is fixed at 10 in the Short Run. Calculate the firm's short run total, average, and marginal cost curves. What is the marginal cost of the 10^{th} unit. The 50^{th} unit? The 100^{th} unit?

If we fix K at 10:
$q = \min(50, 10L)$

So we can't produce more than 50 units of q, and the marginal cost of the 100^{th} unit is ∞.
Below $q = 50$, our production function is simply $q = 10L$, so $L = q/10$

$STC = wq/10 + 10v \quad = \quad 0.3q + 10$
SMC = 0.3 for q up to 50. AT q=50 we have a discontinuity, so SMC = 0.3 going from 50 down, but ∞ going from 50 up.
$SAC = 0.3 + 10/q$ The average cost is declining b/c we are spreading fixed cost.

4. An enterprising entrepreneur purchases two firms to produce widgets. Each firm produces identical products, and each has a production function given by:

$q_i = (K_i L_i)^{1/2}$ Where i = 1, 2 identifies the firm.

The firms differ in the amount of capital equipment each has. $K_1 = 25$, $K_2 = 100$. Rental rates for K and L are given by $v = w = 1$.

A) How should production allocated between the two firms to minimize the short run total costs?

The SRMC must be the same in each firm.

$q_1 = (25L)^{½} = 5L^{1/2}$	$q_2 = (100L)^{½} = 10L^{1/2}$
$L_1 = 25(q_1)^2$	$L_2 = 100(q_2)^2$
$TC_1 = L + 25 = 25(q_1)^2 + 25$	$TC_2 = L + 100 = 100(q_2)^2 + 100$

$P = MC_1 = 50q_1$ $\qquad$ $P = MC_2 = 200q_2$

Set $MC_1 = MC_2$
$50q_1 = 200q_2$
$q_1 = 4q_2$

B) Given that output is optimally allocated between the two firms, calculate the short run total, average, and marginal cost curves. What is the MC of the 100th widget, the 125th widget? The 200th widget?

$q = q_1 + q_2 = 5q_2$

Short Run Total Cost Curve $= L_1(q_1) + L_2(q_2) + 125 = L_1(4q_2) + L_2(q_2) + 125$
$= 25(q_1)^2 + 100(q_2)^2 + 125 = 25(4q/5)^2 + 100(q/5)^2 + 125$
$= 20q^2 + 125$

Short Run Avg Cost Curve $= 20q + 125/q$

Short Run Marginal Cost Curve = 20q

				MC = 40q
For the 100th unit:	$q_2 = 20$;	$q_1 = 80$	MC = 4,000	40*100 = 4000
For the 125th unit:	$q_2 = 25$	$q_1 = 100$	MC = 5,000	40*125 = 5000
For the 200th unit:	$q_2 = 40$	$q_1 = 160$	MC = 8,000	40*200 = 8000
$q_1 = P/50$		$q_2 = P/200$		

C) In the Long Run how should the entrepreneur allocate widget production between the two firms? Calculate the Long Run total, average, and marginal cost curves for widget production.

In the long run, since the two firms have identical CRS cost functions, it doesn't matter how widget production is allocated.

The long run production function is: $q = (K\,L)^{½}$

K = L at equilibrium. q = K = L
TC = L + K = q + q = 2q
MC = 2
AC = 2

D) How would your answer to C) change if both firms exhibited diminishing returns to scale? Increasing returns to scale?

If both firms exhibited diminishing returns to scale, the entrepreneur would split the production equally between the two firms in the long run; if both firms exhibited increasing returns to scale, the entrepreneur would close one firm down and transfer all production to the other.

5) The production function for a firm in the business of widget production is given by:
$q = 2\,L^{1/2}$ where q = number of widgets; L = Labor input (in hours).

The firm is a price taker in output market (Price of widgets = P) and labor market (wage = w).

A) What is the supply function for widgets (q = f(P, w)

$L = q^2/4$ $TC = wL = w(q^2/4)$ $MC = wq/2$

Inverse Supply Function = $P = MC = wq/2$: $q_S = 2P/w$

B) Explain both algebraically and graphically why this supply function is homogenous of degree zero in P and w and why profits are homogenous of degree 1 in these variables.

Homogenous of degree zero in P and w: $q = f(P, w) = f(mP, mw) = 2P/w$

My Mistake – profit is not homogenous of degree one, since revenues ARE degree one, but costs show decreasing returns to scale.

Profits = $g(P, w) = Pq - TC = 2P^2/w - wq^2/4$

$g(mP,mw) = m2P^2/w - m^3wq^2/4$

C) Show explicitly (on a graph) how changes in w shift the supply curve for this firm.

The supply curve: $q_S = 2P/w$

$q_S(w = 1) = 2P$ $q_S(w = 2) = P$ $q_S(w = 3) = 2P/3$ $q_S(w = 4) = P/2$