

Answer Key for Problem Set # 10
Econ 812: Spring 2004

Varian

Chapter 16: 16.1, 16.3, 16.4, 16.10, 16.12.

Chapter 23: 23.6

Chapter 24: 24.1

Overall comments on the problems:

16.1 is easy.

16.3 and 16.4 are more difficult than they ought to be for the amount of intellectual content. If you didn't get them, don't worry about it. (Even I couldn't get the algebra to work on 16.4, although I know how it is supposed to work.)

16.10 – a good review of the basic concepts in oligopoly.

16.12 – this stretches the concepts a little bit, good for insight into applying optimization.

23.6, and 24.1 are more of a stretch, but if you can follow them you are very well prepared. They both exemplify how we can apply the basic concept of defining and optimizing objective functions to lots of different models. 24.1 is very similar to the “city agglomeration” model I distributed earlier in the semester.

16.1

Given 2 firms. Constant MC's c_1 and c_2 , where $c_2 > c_1$. Find Bertrand and Competitive equilibria.

Answer:

Bertrand Equilibrium:

The two firms compete on price, and firm 1 wins.

Price = $c_2 - \epsilon$, where epsilon is some very small number; firm one sells $Q(c_2 - \epsilon)$.

Competitive Equilibrium: Price = c_1 ; firm sells $Q(c_1) > Q(c_2 - \epsilon)$

16.3 and 16.4

First, we need to find the formulas for a_1 , a_2 , b_1 , b_2 , and c .

The Demand Functions (from p. 294):

$$(1) \quad p_1 = \alpha_1 - \beta_1 y_1 - \gamma y_2$$

$$(2) \quad p_2 = \alpha_2 - \gamma y_1 - \beta_2 y_2,$$

can be re-arranged to form:

$$(3) \quad y_1 = a_1 - b_1 p_1 + c p_2$$

$$(4) \quad y_2 = a_2 + c p_1 - b_2 p_2.$$

In order to find the formulas for the parameters – algebra:

Solve equations 1 and 2 for quantities:

$$(1a) \quad y_1 = (\alpha_1 / \beta_1) - (1 / \beta_1) p_1 - (\gamma / \beta_1) y_2$$

$$(1b) \quad y_2 = (\alpha_2 / \beta_2) - (1 / \beta_2) p_2 - (\gamma / \beta_2) y_1$$

And now solve this system y_1 and y_2 :

$$(5a) \quad y_1 = (\alpha_1 \beta_2 - \gamma \alpha_2) / (\beta_1 \beta_2 - \gamma^2) + (\gamma / (\beta_1 \beta_2 - \gamma^2)) p_2 - (\beta_2 / (\beta_1 \beta_2 - \gamma^2)) p_1$$

$$(5b) \quad y_2 = (\alpha_2 \beta_1 - \gamma \alpha_1) / (\beta_1 \beta_2 - \gamma^2) - (\beta_1 / (\beta_1 \beta_2 - \gamma^2)) p_2 + (\gamma / (\beta_1 \beta_2 - \gamma^2)) p_1$$

And if I compare (5a) to 3 and (5b) to 4 I get:

$$a_2 = (\alpha_2 \beta_1 - \gamma \alpha_1) / (\beta_1 \beta_2 - \gamma^2)$$

$$b_2 = (\beta_1 / (\beta_1 \beta_2 - \gamma^2))$$

$$c = (\gamma / (\beta_1 \beta_2 - \gamma^2))$$

$$a_1 = (\alpha_1 \beta_2 - \gamma \alpha_2) / (\beta_1 \beta_2 - \gamma^2)$$

$$b_1 = (\beta_2 / (\beta_1 \beta_2 - \gamma^2))$$

Which are easier to think about if I define $D = (\beta_1 \beta_2 - \gamma^2)$:

$$a_2 = (\alpha_2 \beta_1 - \gamma \alpha_1) / D$$

$$b_2 = (\beta_1 / D)$$

$$c = (\gamma / D)$$

$$a_1 = (\alpha_1 \beta_2 - \gamma \alpha_2) / D$$

$$b_1 = (\beta_2 / D)$$

Problem 16.4

I want to show that the Cournot game yields higher prices and lower quantities than the Bertrand game, even when the two firms are not selling identical products. In order to do this, I will borrow the reaction functions that Varian has already worked out for us (see page 295. Note that I am adopting the assumption in the text that $MC = 0$.)

$$\text{Cournot:} \quad y_1 = (\alpha_1 - \gamma y_2) / 2\beta_1 \qquad y_2 = (\alpha_2 - \gamma y_1) / 2\beta_2$$

$$\text{Bertrand:} \quad p_1 = a_1 + c p_2 / 2b_1 \qquad p_2 = a_2 + c p_1 / 2b_2$$

Prove: The Cournot Solutions (y_i, p_i) have larger y and smaller p than the corresponding Bertrand solutions.

Plan – find the equilibria, compare them.

Finding the equilibria is relatively straightforward, comparing is difficult.

Cournot (solving the reaction functions for y):

$$y_1^* = (2\beta_1 \beta_2 - \gamma \alpha_2) / (4\beta_1 \beta_2 - \gamma^2) \qquad y_2^* = (2\beta_1 \beta_2 - \gamma \alpha_1) / (4\beta_1 \beta_2 - \gamma^2)$$

As you can see, these differ by only a single term.

Bertrand solution (solving the reaction functions for p)

$$p_1^* = (2a_1b_2 + ca_2) / (4b_1b_2 - c^2) \quad p_2^* = (2a_2b_1 + ca_1) / (4b_1b_2 - c^2)$$

As you can see, both terms in the numerator differ.

Now what makes this tricky is that we have quantities for Cournot, and prices for Bertrand. In order to get equilibrium Cournot prices and Bertrand Quantities we need to substitute back into the original demand functions. The problem is that this produces really ugly equations, and their relative sizes are NOT obvious.

Instead – think about the extremes.

I expect when we have perfect substitutes that the Bertrand price will go to the $MC = 0$, while the Cournot price will be greater than zero, and therefore Bertrand quantity will be greater than Cournot quantity.

I expect when we have unrelated products that we have two monopolies – and Bertrand and Cournot results will be the same (since it doesn't matter to a monopolist whether he maximizes on price or quantity.)

In terms of our model, unrelated products means $\gamma = c = 0$. Perfect substitutes means $\beta_1 = \beta_2 = \gamma$.

Perfect substitutes:

If we examine our two equilibriums for perfect substitutes: $\beta_1 = \beta_2 = \gamma = \beta$. We will also measure the goods in the same units, so $\alpha_1 = \alpha_2 = \alpha$

Bertrand – we should get prices going to zero, since $MC = 0$. And we do. Then when price = 0, quantity sold will be $a = \alpha/\beta$.

$$\text{Cournot: } y_1^* = (2\beta_1\beta_2 - \gamma\alpha_2) / (4\beta_1\beta_2 - \gamma^2) = (2\beta^2 - \beta\alpha_2) / (4\beta^2 - \beta^2) = (2/3) - (\alpha_2 / 3\beta)$$

What about unrelated products –

In this case, the two markets will evolve totally separately, and each firm is a monopolist, so we expect the two solutions (Bertrand and Cournot) to both go to the monopoly solution.

Therefore, as long as nothing “weird” happens in between, it must be the case that the Cournot solution has higher prices and lower quantities for any partial substitutes.

16.10

Overall plan – solving game theory problems usually involves FIRST optimizing objective functions, to determine the “reaction function,” then finding the equilibrium that satisfies both (or all) reaction functions simultaneously.

$$P(Y) = 100 - Y \quad Y = y_1 + y_2$$

$$\text{Inverse: } Y = 100 - P$$

$$MC = AC = 0$$

a) Competitive Equilibrium: $P^* = 0, Y^* = 100$.

b) Cournot competition – Firms pick P.

Equilibrium is same as competitive equilibrium, since as long as $P > MC$, we don't have a Nash Equilibrium.

c) Cournot Equilibrium – Firms pick Q.

Think about problem for firm #1: This will give us y_1^* as a function of y_2 , which is known as a reaction function:

$$\text{Profit} = \text{Total Revenue (since } MC = 0) = P(y_1 + y_2) \cdot y_1$$

$$\text{Max } \pi = (100 - y_1 - y_2)y_1 = 100y_1 - y_1^2 - y_1y_2$$

$$d\pi/dy_1 = 100 - 2y_1 - y_2 = 0 \quad y_1^* = 50 - y_2/2$$

Since the problem is symmetric, the reaction function for firm 2 is: $y_2^* = 50 - y_1/2$

And the equilibrium quantity for each firm is then $y_1^* = y_2^* = 100/3 = 33.33\dots$

d) Cartel output = monopoly output

$$\text{Max Total Revenue} = (100 - Y)Y = 100Y - Y^2$$

$$d\pi/dY = 100 - 2Y = 0 \quad Y^* = 50, P = \$2.00$$

e) Stackelberg Problem – see p. 295-296

Firm 1 will choose y_1 first, then Firm 2 will respond by choosing y_2 . So we solve first for firm two's response (since this is what firm 1 will consider when they pick y_1 .)

$$\pi_2 = (100 - y_1 - y_2)y_2 = 100y_2 - y_1y_2 - (y_2)^2$$

The main difference is that firm two can't change firm one's y_1 , so they just pick the best response:

$$d\pi_2/dy_2 = 100 - y_1 - 2y_2 = 0 \quad y_2^* = 50 - (y_1 / 2)$$

And now we go back to the first move and consider firm one's problem:

$$\pi_1 = (100 - y_1 - y_2)y_1 = 100y_1 - (y_1)^2 - y_1y_2$$

and we sub in the solution for y_2^*

$$\begin{aligned}\pi_1 &= 100 y_1 - (y_1)^2 - y_1 (50 - (y_1 / 2)) = 100 y_1 - (y_1)^2 - y_1 (50) + ((y_1)^2 / 2) \\ &= 50 y_1 - ((y_1)^2 / 2)\end{aligned}$$

f.o.c. $d\pi_1/y_1 = 50 - y_1 = 0$ So $y_1^* = 50$; $y_2^* = 50 - 25 = 25$

16.12

a) Supply curve = MC curve for a competitive firm:

$$c(y) = y^2/2 \quad MC = y \quad p = y$$

b) Industry Supply is the Horizontal Sum = $Y_c = 50 * y = 50p$

c) Monopolist Demand = Total Demand – Competitive Supply

$$D^M(p_M) = 1000 - 50p_M - 50p_M = 1000 - 100p_M$$

d, e) monopolists output, price and profit – remember that $MC = 0$ for monopolist.

Monopolist inverse demand: $p_M = 10 - y_M/100$

$$\text{Max } \pi = \text{Max TR} = p * y = (10 - y_M / 100)y_M = 10y_M - y_M^2 / 100$$

$$d\pi_M / dy_M = 10 - y_M / 50 = 0 \quad y_M = 500. \quad p_M = 5$$

f, g) The competitive sector produces $50 * p = 250$ units. Total output = 750.

23.6

The grads have only one good – coconuts (price = 1).

Endowment = ω_i

Private Consumption = x_i

Donation to public good: $g_i = \omega_i - x_i$

Total Public Good: $G = \sum g_i$ and we define $G_{-i} = G - g_i$

Utility of each person is: $u_i = x_i + a_i \ln G$

a) Find the choice that each grad makes. She assumes that G_{-i} , what other people donate, is independent of what she does. (Let's us ignore the reaction function – we don't have a game. One way of thinking about this is that we have a really large department, so each person's activity is not easily observable.)

$$\text{Max } u_i = x_i + a_i \ln G = u_i = x_i + a_i \ln (G_{-i} + \omega_i - x_i)$$

b) Now find the equilibrium:

For individuals who end up donating part of their endowment we can solve the unconstrained maximization problem:

$$\text{f.o.c. } du_i/dx_i = 1 + a_i (-1) / (G_{-i} + \omega_i - x_i) = 0$$

$$\text{therefore } a_i = (G_{-i} + \omega_i - x_i) \quad \text{and} \quad \begin{aligned} x_i^* &= G_{-i} + \omega_i - a_i \\ g_i^* &= a_i - G_{-i} \end{aligned}$$

There are two other kinds of individuals:

1. People who donate everything:

if $g_i^* > \omega_i$, then they are limited to donating ω_i . When will this constraint bind. The characteristics that vary between Grads are ω_i and a_i , so we want to solve for the relationship between ω_i and a_i that satisfies the following inequality:

$$g_i^* = a_i - G_{-i} \geq \omega_i$$

$$a_i - \omega_i \geq G_{-i}$$

Interpretation: Large demand for public good (large a_i) or small endowment both make it more likely that a Grad will max out her donation. So does a small level of participation by the other members of the department.

Extra Exercise – what if this is a department of 1 person. How much G will be produced? Is it Pareto Optimal?

2. People who donate nothing:

Grads who prefer the private good will find themselves consuming $x_i = \omega_i$, and maybe wanting more:

$$\omega_i^* \leq G_{-i} + \omega_i - a_i \quad a_i \leq G_{-i}$$

c) The free-riders are people for whom $a_i \leq G_{-i}$

d) What is the Pareto efficient amount of the public good?

I know the answer has to be that the sum of the marginal benefits from G must equal the MC, and since we are measuring everything in units of coconuts, the MC of taking a coconut out of private hands = 1. Therefore:

$$1 = \sum du_i/dg_i \quad \text{is our equilibrium condition}$$

$$\text{Utility of each person is: } u_i = x_i + a_i \ln G$$

$$\text{Therefore } \partial u_i / \partial g_i = a_i / G \quad \text{note – we are holding } x \text{ constant here.}$$

$$1 = \sum du_i/dg_i = \sum a_i/G = (\sum a_i) / G$$

$$\text{And the Optimal } G^* = (\sum a_i)$$

Note that this suggests the optimal “Lindahl Tax” would be a_i . Also note that this is much larger

than what gets provided, since everyone donates much less than a_i .

24.1

We need to solve for the choice of speed that each individual will choose and the speed that optimizes Social Welfare.

I will assume that the choices of speeds are independent, x_i is not a function of x_{-i} . The problem is a little bit more complex if you let one person adjust to the other person's speed – but it doesn't change the qualitative relationship (each person goes “too fast.”)

I solve for the first person – the problem as it is set up is symmetric:

$$\max_{x_1} U_1 = -p(x_1, x_2) * c_1 + u_1(x_1)$$

$$\text{f.o.c.} \quad dU_1/dx_1 = -(\partial p/\partial x_1) * c_1 + du_1/dx_1 = 0$$

$$\text{Choose } x_i \text{ so that: } (\partial p/\partial x_i) * c_i = du_i/dx_i$$

So each person picks his speed to match $MC = MB$

What choice maximizes Social Welfare? We have to add the costs, but not the benefits – (only one person benefits when he goes faster). So now:

Choose x_1 and x_2 such that:

$$du_1/dx_1 = (\partial p/\partial x_1) * (c_1 + c_2)$$

$$du_2/dx_2 = (\partial p/\partial x_2) * (c_1 + c_2)$$

Now – if we have $u'(x_i) > 0$, but $u''(x_i) < 0$, or declining marginal benefit from speed (or alternately $\partial^2 p/\partial x_i^2 > 0$, then the speed that maximizes personal utility will be greater than the speed that maximizes social utility.

Note that if the c_i and the u_i are different, we could have different optimal speeds for the two drivers.

b) The optimal fine will force each person to face the Social Welfare Maximizing marginal cost. Therefore the fine should equal:

$$\text{fine} = (\partial p/\partial x_i) * c_{-i}$$

and it should be imposed whenever $x_i > x_i^*$ which maximizes social welfare. Note that the fine will increase with speed.

C) Will people ever pay the fines?

If the Social Welfare Maximizer has picked the right speed limits, then we want to charge (fine) any driver who exceeds the limit. But no one will want to exceed the limit, because the fine will exceed the benefit they gain. Therefore the EXPECTED value of what each person loses is the probability of an accident * the cost of an accident:

$$\text{Fine}_1 = p(x_1, x_2) * c_2$$

Note that fines, even if they are charged, do NOT compensate the individual who is suffering from the other guy's speed. The government takes them.

We might prefer a Coasian solution, where each person simply pays the other person for the damage they do. The Coasian solution would start at $x_i > 0$, instead of at $x_i > x_i^*$. Can you write out the total utility function if each person has to pay the other guy for the externality.

d) A more realistic utility function:

$$\max_{x_1} U_1 = -p(x_1, x_2) * (c_1 + u_1(x_1)) + u_1(x_1)$$

f.o.c.
$$dU_1/dx_1 = -(\partial p/\partial x_1) * (c_1 + u_1(x_1)) + du_1/dx_1 = 0$$

$$\text{Choose } x_1 \text{ so that: } (\partial p/\partial x_1) * (c_1 + u_1(x_1)) = du_1/dx_1$$

the fine should now be larger – again, it should be imposed if speed exceeds the social optimum (and hence will never be actually imposed), so now:

$$\text{fine}_1 = p(x_1, x_2) * (c_2 + u_2(x_2)).$$

Note – we force driver 1 to pay for the lost utility of driver 2.